ELSEVIER
Topology and its Applications 123 (2002) 125–129
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

The index of free circle actions in lens spaces

Jan Jaworowski

Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA

Received 11 November 1998; received in revised form 19 August 1999

Abstract

Lens spaces arise from free periodic maps in odd-dimensional spheres. They admit free actions of the circle group. We show that such actions bear a certain resemblance to the standard circle actions in odd-dimensional spheres. In particular, the S^1-indices of such actions are the same as those of the corresponding S^1-actions in the spheres. We note, however, that the constructions of in this paper have no analogue in the case of S^3-actions in lens spaces of dimensions of the form $4n+3$. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: Primary 55M20, Secondary 55M35

Keywords: Lens space; Circle group action; Index; Characteristic class

1. Lens spaces

Definition. Let S^{2n+1} be the unit sphere in the complex $(n+1)$-dimensional space $\mathbb{C}^{n+1}$. Let $e:\mathbb{R}\to S^1$ be the exponential map given by $e(x)=\mathrm{e}^{2\pi x\sqrt{-1}}$ for $x\in\mathbb{R}$. Suppose that p is a natural number. Let $T:\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ be the scalar multiplication by the pth root of one, $e(\frac{1}{p})=\mathrm{e}^{\pm 2\pi\sqrt{-1}/p}z$; i.e., $T(z)=e(\frac{1}{p})z$ for $z\in S^{2n+1}$. The map T generates the standard complex representation of the cyclic group $\mathbb{Z}_p$. It is a complex unitary map of period p acting freely on S^{2n+1}. The orbit space $L_p^{2n+1}=S^{2n+1}/T$ is *the lens space* (corresponding to the period p); it is a closed $(2n+1)$-manifold. In particular, if $p=2$ then L_2^{2n+1} is just the real projective $(2n+1)$-space P^{2n+1}.

Remark. The lens spaces are usually understood in a more general sense by allowing T to act through different powers of $\mathrm{e}^{2\pi\sqrt{-1}/p}$ on different coordinates of z (see [1]). The homology groups of the resulting spaces depend only on p, but their homotopy classification is more complicated. They provide good examples of simple homotopy

E-mail address: jaworows@indiana.edu (J. Jaworowski).

0166-8641/02/$ – see front matter
PII: S0166-8641(01)00175-4

equivalence. In this paper we confine ourselves to the special case of L_p^{2n+1}. In this case the map T has a geometric meaning being just scalar multiplication. This fact is relevant in extending the results of this note to bundles of lens spaces.

In [5] we studied free involutions in lens spaces. We showed that if p is odd then their indices with coefficients in $\mathbb{Z}_2$ were the same as those of the antipodal involution in S^{2n+1}. In the present paper we carry out a similar study free S^1 actions in lens spaces and in bundles of lens spaces. We show that the indices of these actions are the same as those of the standard circle actions in S^{2n+1}. Unlike in the case of involutions in L_p^{2n+1}, the parity of p for circle actions is not an issue, so our results apply, in particular, to real projective spaces.

Referee suggestions and comments on the original version of the paper have led to a considerable simplification of some arguments in Section 3. Those suggestions are gratefully acknowledged.

2. A circle group action in lens spaces

The scalar multiplication of $z \in S^{2n+1}$ by $e(\frac{x}{p})$, where $x \in \mathbb{R}$, defines a group homomorphism $g : \mathbb{R} \to \operatorname{Aut}(S^{2n+1})$ which commutes with the periodic map $T : S^{2n+1} \to S^{2n+1}$. Thus g defines a homomorphism $\bar{g} : \mathbb{R} \to \operatorname{Aut}(L_p^{2n+1})$, i.e., an action of $\mathbb{R}$ on L_p^{2n+1}. If $x \in \mathbb{R}$ is an integer, then $e(\frac{x}{p}) = (e(\frac{1}{p}))^x$, hence the integers in $\mathbb{R}$ act trivially on L_p^{2n+1}. Therefore $\bar{g}$ factors through the exponential map $e : \mathbb{R} \to S^1$ and so it defines a homomorphism $h : S^1 \to \operatorname{Aut}(L_p^{2n+1})$, i.e., an action of S^1 on L_p^{2n+1}. Explicitly this action is defined as follows. If $s \in S^1$ and $x \in \mathbb{R}$ is such that $s = e(x)$ then $s \cdot [z] := [e(\frac{x}{p})z]$, where $z \in S^{2n+1}$ and $[z]$ is the orbit of z in L_p^{2n+1}.

Proposition. *The S^1-action in L_p^{2n+1} defined above is free.*

Proof. Suppose that $s \in S^1$, $[z] \in L_p^{2n+1}$, and $s \cdot [z] = [z]$. Let $s = e(x)$, where $x \in \mathbb{R}$. Then $e(\frac{x}{p})z = (e(\frac{1}{p}))^k z$ for some integer k. Since at least one coordinate of z must be different from zero, it follows that $e(\frac{x}{p}) = (e(\frac{1}{p}))^k$, i.e., $e(\frac{x}{p}) = e(\frac{k}{p})$. This implies that $x/p - k/p$ is an integer, say, $(x-k)/p = m \in \mathbb{Z}$. Thus $x = mp + k$ is an integer and $s = e(x) = 1$. $\square$

Suppose that G is a group acting on a space X. We will denote by $\overline{X} = X/G$ the orbit space of the action.

Lemma. *The orbit space $\bar{L}_p^{2n+1}$ of the S^1-action in L_p^{2n+1} is homeomorphic to the complex projective n-space $P_n\mathbb{C}$.*

Proof. Consider the orbit maps $\rho : S^{2n+1} \to P_n\mathbb{C}$, $\alpha : S^{2n+1} \to L_p^{2n+1}$ and $\pi : L_p^{2n+1} \to \bar{L}_p^{2n+1}$. The identifications $\rho : S^{2n+1} \to P_n\mathbb{C}$ and $\alpha : S^{2n+1} \to L_p^{2n+1}$ are given by multiplication by a complex number (by a complex number of norm one or by a pth root of one). The composition $\pi\alpha : S^{2n+1} \to \bar{L}_p^{2n+1}$ maps each fibre of ρ to a single point and thus defines a map $\beta : \bar{L}_p^{2n+1} \to P_n\mathbb{C}$. It is also easy to check that β is bijective, and thus a homeomorphism. $\square$

3. The index of the S^1-action

We will be using cohomology of the Čech type with coefficients in a ring R. For the most part of the paper, R will be the ring $\mathbb{Z}$ of integers, but we will also make some comments in the case when R is the ring $\mathbb{Q}$ of rational numbers. The Čech cohomology has a continuity property which says that if a cohomology class vanishes on a closed set, then it vanishes on a neighborhood of this set.

The concept of index of a group action on a space was first defined by Yang in [6] for free involutions. The Yang index is an integer; it is the cup-height of the characteristic class of the involution. The concept of index was later extended to more general group actions. Fadell and Husseini [2] defined the index for an arbitrary action of a compact Lie group G on a space; in this generalized sense the index is an ideal in the cohomology ring of the classifying space of G (compare also [4]).

An integer-valued index can be defined in a similar way for other groups whose classifying space is similar to that of $G = \mathbb{Z}_2$. For free circle actions it is defined as follows. A space X with a free action of S^1 is a principal S^1-fibre bundle over the orbit space $\overline{X} = X/S^1$. The universal bundle for S^1 has the infinite dimensional sphere S^∞ as the total space, with the standard S^1-action. Its orbit space, the infinite complex projective space $P_\infty\mathbb{C}$, is a classifying space for S^1. The cohomology ring $H^*P_\infty\mathbb{C}$ is isomorphic to $\mathbb{Z}[c]$, where $c \in H^2(P_\infty\mathbb{C})$ is the characteristic class of the bundle $S^\infty \to P_\infty\mathbb{C}$, i.e., the 1st universal Chern class. If X is space with a free action of S^1, we will denote by $c_X \in H^2\overline{X}$ the characteristic class of the action; i.e., c_X is the image of $c \in H^2 P_\infty\mathbb{C}$ under the map $H^*P_\infty\mathbb{C} \to H^*\overline{X}$ induced by a classifying map $\overline{X} \to P_\infty\mathbb{C}$. The index, $\mathrm{Ind}^{S^1}(X)$, is the largest integer n (if it exists) such that $(c_X)^n \neq 0$. Thus the index of S^{2n+1} with the standard circle action (defined by scalar multiplication) is n.

Remark. The standard scalar multiplication action of $\mathbb{Z}_2 \cong S^0$ on spheres S^n (the real case) and of S^1 on odd-dimensional spheres S^{2n+1} (the complex case) have their analogue in the quaternionic case; it is the scalar multiplication action of S^3 on spheres of dimension $4n+3$. The classifying space for S^3 is the infinite quaternionic projective space $P_\infty\mathbb{H}$. The cohomology ring $H^*P_\infty\mathbb{H}$ is isomorphic to $\mathbb{Z}[p]$, where $p \in H^4P_\infty\mathbb{H}$ is the 1st universal Pontryagin class. The index of an S^3-action can be defined just as the index of an S^1-action (comp. [3]). It might seem, therefore, that the ideas of this paper might be extended to actions of S^3 on lens spaces of dimensions $4n+3$. However, this is not so since, in contrast to the real and complex cases, there is no free action of S^3 on such lens spaces, even in the case of $n = 1$. Indeed, a free action of S^3 on L_p^3 would have just one orbit which would be homeomorphic to S^3; but $L_p^3 \not\cong S^3$. It is just a coincidence that $L_p^1 \cong S^1$.

To simplify the notation we will write $c_L := c_{L_p^{2n+1}}$.

Theorem 1.

(1) *For $R = \mathbb{Z}$, the characteristic class of the bundle $\pi : L_p^{2n+1} \to P_n\mathbb{C}$ is $c_L = p \cdot e_2$, where $e_2 \in H^{2n}P_n\mathbb{C}$ is the generator*; *and* $\mathrm{Ind}^{S^1}(L_p^{2n+1}) = n$.

(2) *For $R = \mathbb{Q}$, the Gysin sequence of the bundle $\pi : L_p^{2n+1} \to P_n\mathbb{C}$ is isomorphic to that of the Hopf bundle $\pi : S^{2n+1} \to P_n\mathbb{C}$ and so the rational S^1-index of L_p^{2n+1} is also n.*

Proof. If $\pi : X \to \overline{X}$ is a principal S^1-bundle then the Gysin sequence of p is as follows:

$$\begin{aligned} &0 \longrightarrow H^1\overline{X} \xrightarrow{\pi^*} H^1X \xrightarrow{\tau} H^0\overline{X} \xrightarrow{\cdot c_X} H^2\overline{X} \xrightarrow{\pi^*} \cdots \\ &\xrightarrow{\cdot c_X} H^{j-1}\overline{X} \xrightarrow{\pi^*} H^{j-1}L_p \xrightarrow{\tau} H^{j-2}\overline{X} \xrightarrow{\cdot c_X} H^j\overline{X} \xrightarrow{\pi^*} H^jX \xrightarrow{\tau} \cdots. \end{aligned}$$

In this sequence $\cdot c_X$ is the cup product with the characteristic class and τ is the "integration along the fibre".

The homology groups of the lens spaces are:

$$H_0L_p^{2n+1} \cong H_{2n+1}L_p^{2n+1} \cong \mathbb{Z}, \qquad H_{2i-1}L_p^{2n+1} \cong \mathbb{Z}_p \quad \text{for } 0 < i \leqslant n;$$

and $H_jL_p^{2n+1} = 0$ in the remaining dimensions (see [1,5]). Thus the cohomology groups of L_p^{2n+1} (with integer coefficients) are

$$\begin{aligned} &H^0L_p^{2n+1} \cong H^{2n+1}L_p^{2n+1} \cong \mathbb{Z}, \\ &H^{2i}L_p^{2n+1} \cong \mathrm{Ext}\big(H_{2i-1}L_p^{2n+1}, \mathbb{Z}\big) \cong \mathrm{Ext}(\mathbb{Z}_p, \mathbb{Z}) \cong \mathbb{Z}_p \quad \text{for } 0 < i \leqslant n, \end{aligned}$$

and are zero in the remaining dimensions.

For the bundle $\pi : L_p^{2n+1} \to P_n\mathbb{C}$ with $\overline{X} = P_n\mathbb{C}$, the Gysin sequence begins with zeroes, breaks up into short exact sequences

$$0 \xrightarrow{\tau} H^{2i-2}P_n\mathbb{C} \xrightarrow{\cdot c_L} H^{2i}P_n\mathbb{C} \xrightarrow{\pi^*} H^{2i}P_n\mathbb{C} \xrightarrow{\tau} 0, \quad 1 < i \leqslant n,$$

and ends with $0 \xrightarrow{\pi^*} H^{2n+1}L_p^{2n+1} \overset{\tau}{\cong} H^{2n}P_n\mathbb{C} \to 0$. The short exact sequences in the middle are isomorphic to $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}_p \to 0$, the second arrow being multiplication by p. This proves part (1) of the theorem.

In the same Gysin sequence, but written with rational coefficients, $H^jL_p^{2n+1} = 0$ for all $0 < j < 2n+1$, so that the sequence reduces to a sequence of isomorphisms $\cdot c_L : H^{2i-2}P_n\mathbb{C} \cong H^{2i}P_n\mathbb{C}$. This proves part (2). $\square$

4. Maps of lens spaces into representation spaces for S^1

The $\mathbb{Z}_2$-index of a space X with an involution $\alpha : X \to X$ is used in proving Borsuk–Ulam–Yang type theorems. Given a map $f : X \to \mathbb{R}^k$ of X into a Euclidean space $\mathbb{R}^k$, such theorems give information about the size of *the antipodal coincidence set* $A(f) = \{x \in X \mid f(\alpha(x)) = f(x)\}$.

If G is a compact Lie group acting on a space X, if V is a representation space for G and if $f : X \to V$ is a equivariant map then the index of X can be used to obtain information on the set of zeroes of f. If the map f is not necessarily equivariant, we can use an averaging construction to replace it by an equivariant map (compare [3]): for any map $f : X \to V$ we define an equivariant map $Avf : X \to V$ so that $Avf = f$ if f is equivariant. Extending the concept of antipodal coincidence set $A(f)$ for spaces with involutions, we define $A(f) = (Avf)^{-1}(0)$. In other words, $A(f)$ is the union of the orbits where the average of f is zero ("balanced orbits", comp. [3]). In particular, for lens spaces L_p^{2n+1} with S^1-actions defined above we obtain results analogous to those which hold for the standard circle actions on odd-dimensional spheres.

Borsuk–Ulam–Yang Theorem. *Suppose that $f : L_p^{2n+1} \to \mathbb{C}^k$ is a map. Then*

$$\mathrm{Ind}^{S^1} A(f) \geqslant n - k.$$

In particular, in terms of covering dimension, $\dim A(f) = \dim \overline{A(f)} + 1 \geqslant 2(n-k) + 1$.

References

[1] P.M. Cohen, A Course in Simple Homotopy Theory, Graduate Texts in Math., Vol. 10, Springer, Berlin, 1970.

[2] E. Fadell, S. Husseini, An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems, Ergodic Theory Dynamical Systems 8 (1988) 73–85.

[3] J. Jaworowski, The set of balanced points of S^1 and S^3 actions, Proc. Amer. Math. Soc. 98 (1986) 158–162.

[4] J. Jaworowski, Maps of Stiefel manifolds and a Borsuk–Ulam theorem, Proc. Edinburgh Math. Soc. (1989) 271–279.

[5] J. Jaworowski, Involutions in lens spaces, Topology Appl. 94 (1999) 155–162.

[6] C.T. Yang, On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobô and Dyson, I, Ann. of Math. (2) 60 (1954) 262–282.